

EPSILON COHERENT STATES WITH POLYANALYTIC COEFFICIENTS FOR THE HARMONIC OSCILLATOR

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ABSTRACT. We construct a new class of coherent states indexed by points z of the complex plane and depending on two positive parameters m and $\varepsilon > 0$ by replacing the coefficients $z^n/\sqrt{n!}$ of the canonical coherent states by polyanalytic functions. These states solve the identity of the states Hilbert space of the harmonic oscillator at the limit $\varepsilon \rightarrow 0^+$ and obey a thermal stability property. Their wavefunctions are obtained in a closed form and their associated Bargmann-type transform is also discussed.

1. INTRODUCTION

In general, coherent states (CS) are a specific overcomplete family of vectors in the Hilbert space of the problem that describes the quantum phenomena and solves the identity of this Hilbert space. These states have long been known for the harmonic oscillator (HO) potential and their properties have frequently been taken as models for defining this notion for other models [1]-[2]. The HO potential finds application in the description of vibrational modes in nuclei, atoms, molecules and crystal lattices.

Here, the CSs for HO potential we are introducing are quite different from the existing ones in the above literature and are simply obtained by adopting a general Hilbertian probabilistic scheme [3] reminiscent to the classical construction of the Bargmann transform [22]. Our procedure can be described as follows. In [5] we have introduced a family of CS for the HO potential through superpositions of the corresponding eigenstates where the role of coefficients $z^n/\sqrt{n!}$ of the canonical CS was played by coefficients

$$(1.1) \quad \Phi_n^m(z) := (-1)^{m \wedge n} (m \wedge n)! |z|^{m-n} e^{-i(m-n) \arg z} L_{m \wedge n}^{(|m-n|)}(z\bar{z}), z \in \mathbb{C}, n = 0, 1, \dots,$$

where $L_n^{(\alpha)}(\cdot)$ denotes the Laguerre polynomial [14] and $m \wedge n = \min(m, n)$. To be more precise, the $\{\Phi_n^m(z)\}$ constitute an orthonormal basis of a true polyanalytic space attached to a fixed m th Landau level [4]. Here, we proceed by modifying the coefficients (1.1) by a factor $e^{-n\varepsilon}$. This defines what we call *epsilon* coherent states and we denote by ε -CS for brevity. In fact these ε -CS solve an ε -identity operator which has the advantage of being a compact and trace class operator. The latter becomes the identity operator of the Hilbert space $L^2(\mathbb{R})$ at the limit $\varepsilon \rightarrow 0^+$. This can be proved essentially by using a result on the Poisson kernel for Hermite polynomials, which is due to Muckenhoopt [11].

On the physical side, we can interpret the number ε as the usual parameter $\beta = 1/k_B T$ of statistical physics where k_B is the Boltzmann constant and T is the temperature. Therefore the resolution of the ε -identity operator gives in fact (up to a normalization factor) the thermodynamical quantum density for a HO potential with the form $\hat{\rho} = \sum_{n=0}^{\infty} n |n\rangle \langle n|$. These ε -CS also obey a thermal stability property. Furthermore, the method we are using, which is similar to the one used in [15], makes possible to obtain a closed form for the ε -CS allowing to define a Bargmann-type transform, say B_m^ε . The latter one can be considered as generalization with the respect to parameter ε of the true-polyanalytic m -Bargmann transform [7]-[4].

The paper is organized as follows. In Section 2, we recall briefly some needed fact on polyanalytic functions on the complex plane. Section 3 deals with the coherent states formalism we will be using. In section 4 we construct ε -coherent states and we show that they solve an ε -identity which becomes the identity of the states Hilbert space at the limit $\varepsilon \rightarrow 0^+$. In Section 5, we give a closed form for the constructed states and we discuss their associated Bargmann-type transform.

2. POLYANALYTIC FUNCTIONS ON \mathbb{C}

The Bargmann-Fock space $\mathbf{F}^{m+1}(\mathbb{C})$ of *polyanalytic* functions consists of all functions $F(z)$ satisfying the equation

$$(2.1) \quad \left(\frac{\partial}{\partial \bar{z}} \right)^{m+1} F(z) = 0$$

and such that

$$(2.2) \quad \int_{\mathbb{C}} |F(z)|^2 e^{-z\bar{z}} d\mu(z) < +\infty$$

where $d\mu(z)$ denotes the Lebesgue measure on \mathbb{C} . Functions satisfying (2.1) are known as polyanalytic functions of order $m+1$. Since Eq.(2.1) generalizes the Cauchy-Riemann equation

$$(2.3) \quad \frac{\partial}{\partial \bar{z}} F(z) = 0,$$

then the space $\mathbf{F}^{m+1}(\mathbb{C})$ is a generalization of the well known Bargmann-Fock space $\mathcal{F}(\mathbb{C})$ of entire Gaussian-square integrable functions on \mathbb{C} . That is, for $m = 0$, $\mathbf{F}^1(\mathbb{C}) \equiv \mathcal{F}(\mathbb{C})$, see [12], [4]. Polyanalytic functions inherit some of the properties of analytic functions, in a nontrivial form. However, many of the properties break down once we leave the analytic setting. For instance, while nonzero entire functions do not have sets of zeros with an accumulation point, polyanalytic functions can vanish along closed curves. To illustrate such situation, take $F(z) := z\bar{z} - 1$, a polyanalytic function of order 2.

Now, if we look at the so-called *true* polyanalytic Fock spaces [7], [8] which will be denoted here by $\mathcal{A}_l^2(\mathbb{C})$, $l = 0, 1, \dots, m$. These spaces are related to the polyanalytic Fock space $\mathbf{F}^{m+1}(\mathbb{C})$ by the orthogonal decomposition ([4], [7], [8]):

$$(2.4) \quad \mathbf{F}^{m+1}(\mathbb{C}) = \mathcal{A}_0^2(\mathbb{C}) \oplus \mathcal{A}_1^2(\mathbb{C}) \oplus \dots \oplus \mathcal{A}_m^2(\mathbb{C}).$$

Moreover, for each fixed $m \in \mathbb{Z}_+$, the true polyanalytic Bargmann-Fock space $\mathcal{A}_m^2(\mathbb{C})$ admits a nice realization as an eigenspace ([13]):

$$(2.5) \quad \mathcal{A}_m^2(\mathbb{C}) := \left\{ f \in L^2(\mathbb{C}, e^{-z\bar{z}} d\mu), \tilde{\Delta} f = m f \right\}$$

of the second order differential operator

$$(2.6) \quad \tilde{\Delta} := -\frac{\partial^2}{\partial z \partial \bar{z}} + \bar{z} \frac{\partial}{\partial \bar{z}}.$$

This operator constitutes, in suitable units and up to additive constant, a realization in the Hilbert space $L^2(\mathbb{C}, e^{-z\bar{z}} d\mu)$ of the Schrödinger operator with uniform magnetic field in \mathbb{C} . Its spectrum consists of eigenvalues λ_m of infinite multiplicity (*Euclidean Landau levels*) of the form $\lambda_m := m$, $m \in \mathbb{Z}_+$. The space in (2.5) admits an orthogonal basis whose elements are expressed by

$$(2.7) \quad \Phi_n^m(z) := (-1)^{m \wedge n} (m \wedge n)! |z|^{m-n} e^{-i(m-n) \arg z} L_{m \wedge n}^{(|m-n|)}(z\bar{z}), z \in \mathbb{C}, n = 0, 1, \dots,$$

in terms of Laguerre polynomials [19] :

$$(2.8) \quad L_j^{(\alpha)}(t) = \sum_{k=0}^j \frac{\Gamma(\alpha + j + 1)}{\Gamma(\alpha + k + 1)} \frac{(-t)^k}{(j-k)!k!}, \quad \alpha > -1$$

with the orthogonality relations (with respect to the scalar product in $L^2(\mathbb{C}, e^{-z\bar{z}} d\mu)$) given by

$$(2.9) \quad \langle \Phi_n^m | \Phi_j^m \rangle = \pi m! n! \delta_{n,j} = \pi m! n! \delta_{n,j}.$$

where $\delta_{j,k}$ denotes the Kronecker symbol. Direct calculations using (2.7),(2.9) together with a known summation formula of the product of Laguerre polynomials allow to obtain the reproducing kernel of the Hilbert space $\mathcal{A}_m^2(\mathbb{C})$ with the form ([16]) :

$$(2.10) \quad K_m(z, w) = \pi^{-1} e^{z\bar{w}} L_m^{(0)}(|z-w|^2), \quad z, w \in \mathbb{C}.$$

More information on these spaces and applications to signal analysis and physics can be found in [4], [6], [10] and references therein.

3. EPSILON COHERENT STATES

In this section, we will review a generalization of canonical CS by considering a kind of the identity resolution that we obtain at the zero limit with respect to a parameter $\varepsilon > 0$. Their formalism can be found in [15] where new families of CS attached to the Hamiltonian with pseudo-harmonic oscillator potential were constructed.

Definition 3.1 . Let \mathcal{H} be a (complex, separable, infinite-dimensional) Hilbert space with an orthonormal basis $\{\psi_n\}_{n=0}^\infty$. Let $\mathfrak{D} \subseteq \mathbb{C}$ be an open subset of \mathbb{C} and let $c_n : \mathfrak{D} \rightarrow \mathbb{C}; n = 0, 1, 2, \dots$, be a sequence of complex functions. Define

$$(3.1) \quad |z, \varepsilon\rangle := (\mathcal{N}_\varepsilon(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\overline{c_n(z)}}{\sqrt{\sigma_\varepsilon(n)}} |\psi_n\rangle$$

where $\mathcal{N}_\varepsilon(z)$ is a normalization factor and $\sigma_\varepsilon(n); n = 0, 1, 2, \dots$, a sequence of positive numbers depending on $\varepsilon > 0$. The vectors $\{|z, \varepsilon\rangle, z \in \mathfrak{D}\}$ are said to form a set of epsilon coherent states if

- (i) for each fixed $z \in \mathfrak{D}$ and $\varepsilon > 0$, the state in (3.1) is normalized, that is $\langle z, \varepsilon | z, \varepsilon \rangle_{\mathcal{H}} = 1$,
(ii) the following resolution of the identity operator on \mathcal{H}

$$(3.2) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\mathfrak{D}} |z, \varepsilon\rangle \langle z, \varepsilon| d\mu_{\varepsilon}(z) = \mathbf{1}_{\mathcal{H}}$$

is satisfied with an appropriately chosen measure $d\mu_{\varepsilon}$.

In the above definition, the Dirac's *bra-ket* notation $|z, \varepsilon\rangle \langle z, \varepsilon|$ in (3.2) means the rank-one operator $\varphi \mapsto |z, \varepsilon\rangle \langle z, \varepsilon| \varphi\rangle_{\mathcal{H}}$, $\varphi \in \mathcal{H}$. Also, the limit in (3.2) is to be understood as follows. We define the integral of rank-one operators as being the linear operator

$$(3.3) \quad \mathcal{O}_{\varepsilon}[\varphi](\bullet) := \int_{\mathfrak{D}} \langle \bullet | z, \varepsilon \rangle \langle z, \varepsilon | \varphi \rangle d\mu_{\varepsilon}(z).$$

Then, the above limit is pointwise meaning $\mathcal{O}_{\varepsilon}[\varphi](\bullet) \rightarrow \varphi(\bullet)$ as $\varepsilon \rightarrow 0^+$, *almost every where* with respect to (\bullet) . Here, we should mention that the usual way is to understand the integral

$$(3.4) \quad \int_{\mathfrak{D}} |z, \varepsilon\rangle \langle z, \varepsilon| d\mu_{\varepsilon}(z)$$

in the weak sense, see for instance ([5], p.8). Namely, it is the sesquilinear form

$$(3.5) \quad B_{\varepsilon}(\phi, \psi) := \int_{\mathfrak{D}} \langle \phi | z, \varepsilon \rangle \langle z, \varepsilon | \psi \rangle d\mu_{\varepsilon}(z).$$

Choosing this way, one has to check that the form (3.5) is bounded so that the Riesz lemma ensures the existence of a unique bounded operator $\mathcal{O}_{\varepsilon}$ satisfying $B_{\varepsilon}(\phi, \psi) = \langle \phi | \mathcal{O}_{\varepsilon}[\psi] \rangle$. In our framework the resolution of the identity reads $\lim_{\varepsilon \rightarrow 0} B_{\varepsilon}(\phi, \psi) = \langle \phi | \psi \rangle$ meaning that $\lim_{\varepsilon \rightarrow 0} \mathcal{O}_{\varepsilon} = \mathbf{1}_{\mathcal{H}}$ in the weak operator topology.

Note also that the above expression (3.1) can be viewed as a generalization of the series expansion of the canonical (anti-holomorphic) coherent states

$$(3.6) \quad |z\rangle := (e^{z\bar{z}})^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\bar{z}^n}{\sqrt{n!}} |\varphi_n\rangle; \quad z \in \mathbb{C},$$

where $\{|\varphi_n\rangle\}$ is an orthonormal basis in $L^2(\mathbb{R})$, which consists of eigenstates of the Hamiltonian with the HO potential $-\partial_x^2 + x^2$ given by

$$(3.7) \quad \varphi_n(x) = (\sqrt{\pi} 2^n n!)^{-1/2} e^{-\frac{1}{2}x^2} H_n(x)$$

in terms of the Hermite polynomial

$$(3.8) \quad H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k! (n-2k)!} (2x)^{n-2k}$$

see [19]. Here, the notation $\lfloor a \rfloor$ means the greatest integer not exceeding a .

4. EPSILON CS WITH POLYANALYTIC COEFFICIENTS FOR THE HO POTENTIAL

We now construct a class of ε -CS indexed by points $z \in \mathbb{C}$ and depending on two parameters m and ε by replacing the coefficients $z^n/\sqrt{n!}$ of the canonical coherent states by polyanalytic coefficients as mentioned in the introduction.

Definition 4.1. Define a set of states labeled by points $z \in \mathbb{C}$ and depending on two parameters m and $\varepsilon > 0$ by the following superposition

$$(4.1) \quad |z; m, \varepsilon\rangle := (\mathcal{N}_{m, \varepsilon}(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\overline{\Phi_n^m(z)}}{\sqrt{\sigma_{\varepsilon, m}(n)}} |\varphi_n\rangle$$

where $\mathcal{N}_{m, \varepsilon}(z)$ is a normalization factor, $\sigma_{m, \varepsilon}(n)$ are positive numbers given by

$$(4.2) \quad \sigma_{m, \varepsilon}(n) := \pi m! n! e^{n\varepsilon}, \quad n = 0, 1, 2, \dots,$$

and $\{|\varphi_n\rangle\}$ is the orthonormal basis of $\mathcal{H} = L^2(\mathbb{R})$, consisting of eigenstates of the harmonic oscillator as given in (3.7).

In the next result (see Appendix A) we give the overlap relation between two ε -CS.

Proposition 4.1. Let $m \in \mathbb{Z}_+$ and $\varepsilon > 0$. Then, for every z, w in \mathbb{C} , the overlap relation between two ε -CS is expressed as

$$(4.3) \quad \langle z; m, \varepsilon | w; m, \varepsilon \rangle_{L^2(\mathbb{R})} = \frac{\exp(e^{-\varepsilon} z \bar{w} - m\varepsilon)}{\pi \sqrt{\mathcal{N}_{m, \varepsilon}(z) \mathcal{N}_{m, \varepsilon}(w)}} L_m^{(0)}((ze^{-\varepsilon} - w)(\bar{z}e^{\varepsilon} - \bar{w}))$$

where the normalization factor is given by

$$(4.4) \quad \mathcal{N}_{m, \varepsilon}(z) = \pi^{-1} \exp(e^{-\varepsilon} z \bar{z} - m\varepsilon) L_m^{(0)}(2(1 - \cosh \varepsilon) z \bar{z})$$

in terms of the Laguerre polynomial $L_m^{(0)}(\cdot)$.

Corollary 4.1. At the limit $\varepsilon \rightarrow 0^+$, the overlap relation (4.3) gives the normalized reproducing kernel of the true polyanalytic Fock space $\mathcal{A}_m^2(\mathbb{C})$. That is,

$$(4.5) \quad \lim_{\varepsilon \rightarrow 0^+} \langle z; m, \varepsilon | w; m, \varepsilon \rangle_{L^2(\mathbb{R})} = \frac{K_m(z, w)}{\sqrt{K_m(z, z) K_m(w, w)}}$$

where $K_m(z, w)$ is given explicitly by (2.10).

We now proceed to determine a measure of the form $\mathcal{N}_{m, \varepsilon}(z) d\eta(z)$ with respect to which the ε -CS satisfy a resolution of an ε -identity operator and where $d\eta(z)$ is not ε -independent.

Proposition 4.2. The ε -CS solve an ε -identity operator as follows

$$(4.6) \quad {}_{\mathbb{C}} |z; m, \varepsilon\rangle \langle z; m, \varepsilon| d\mu_{m, \varepsilon}(z) = e^{-\varepsilon \mathbf{H}}$$

where $\mathbf{H} = \sum_{n=0}^{+\infty} n |\varphi_n\rangle \langle \varphi_n|$ and

$$(4.7) \quad d\mu_{m, \varepsilon}(z) = e^{-z\bar{z}} \pi^{-1} \exp(e^{-\varepsilon} z \bar{z} - m\varepsilon) L_m^{(0)}(2(1 - \cosh \varepsilon) z \bar{z}) d\mu(z)$$

with $d\mu(z)$ being the Lebesgue measure on \mathbb{C} .

Proof. Let us assume that the measure takes the form

$$(4.8) \quad d\mu_{m,\varepsilon}(z) = \mathcal{N}_{m,\varepsilon}(z) \rho(z) d\mu(z)$$

where $\rho(z)$ is an auxiliary density to be determined. Let $\varphi \in L^2(\mathbb{R})$ and let us start according to (3.3) by writing

$$(4.9) \quad \mathcal{O}_{m,\varepsilon}[\varphi] := {}_{\mathbb{C}}\langle z; m, \varepsilon \rangle \langle z; m, \varepsilon | d\mu_{m,\varepsilon}(z) \rangle [\varphi]$$

$$(4.10) \quad = {}_{\mathbb{C}}\langle \varphi | z; m, \varepsilon \rangle \langle z; m, \varepsilon | d\mu_{m,\varepsilon}(z)$$

$$(4.11) \quad = \int_{\mathbb{C}} \langle \varphi | (\mathcal{N}_{m,\varepsilon}(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\overline{\Phi_n^m(z)}}{\sqrt{\sigma_{\varepsilon,m}(n)}} |\varphi_n\rangle \rangle \langle z; m, \varepsilon | d\mu_{m,\varepsilon}(z)$$

$$(4.12) \quad = \int_{\mathbb{C}} \sum_{n=0}^{+\infty} \frac{\overline{\Phi_n^m(z)}}{\sqrt{\sigma_{\varepsilon,m}(n)}} \langle \varphi | \varphi_n \rangle \langle z; m, \varepsilon | (\mathcal{N}_{m,\varepsilon}(z))^{-\frac{1}{2}} d\mu_{m,\varepsilon}(z)$$

$$(4.13) \quad = \left(\sum_{n,j=0}^{+\infty} \int_{\mathbb{C}} \frac{\overline{\Phi_n^m(z)} \Phi_j^m(z)}{\sqrt{\sigma_{\varepsilon,m}(n)} \sqrt{\sigma_{\varepsilon,m}(j)}} |\varphi_n\rangle \langle \varphi_j | (\mathcal{N}_{m,\varepsilon}(z))^{-1} d\mu_{m,\varepsilon}(z) \right) [\varphi].$$

Replace $d\mu_{m,\varepsilon}(z) = \mathcal{N}_{m,\varepsilon}(z) \rho(z) d\mu(z)$, then Eq.(4.13) takes the form

$$(4.14) \quad \mathcal{O}_{m,\varepsilon} = \sum_{n,j=0}^{+\infty} e^{-(n+j)\frac{\varepsilon}{2}} \left[\int_{\mathbb{C}} \frac{\Phi_j^m(z) \overline{\Phi_n^m(z)}}{\sqrt{\pi m! j!} \sqrt{\pi m! n!}} \rho(z) d\mu(z) \right] |\varphi_n\rangle \langle \varphi_j|.$$

We recall the orthogonality relations (2.9) :

$$(4.15) \quad \langle \Phi_n^m | \Phi_j^m \rangle = \pi m! n! \delta_{n,j}.$$

This suggests us to set

$$(4.16) \quad \rho(z) := e^{-z\bar{z}}, z \in \mathbb{C}.$$

Therefore, the operator in (4.14) takes the form

$$(4.17) \quad \mathcal{O}_{m,\varepsilon}[\varphi] \equiv \mathcal{O}_{\varepsilon}[\varphi] = \sum_{n=0}^{+\infty} e^{-n\varepsilon} (|\varphi_n\rangle \langle \varphi_n|) [\varphi].$$

By defining a Hamiltonian operator of the harmonic oscillator type via the discrete spectral resolution $\mathbf{H} = \sum_{n=0}^{+\infty} n |\varphi_n\rangle \langle \varphi_n|$, then Eq.(4.14) also reads $\mathcal{O}_{\varepsilon}[\varphi] = e^{-\varepsilon \mathbf{H}}[\varphi]$, $\varphi \in \mathcal{H}$. \square

In the next result (see Appendix B) we state the resolution of the identity operator.

Proposition 4.3. *The ε -CS satisfy the following resolution of the identity*

$$(4.18) \quad \lim_{\varepsilon \rightarrow 0^+} {}_{\mathbb{C}}\langle z; m, \varepsilon \rangle \langle z; m, \varepsilon | d\mu_{m,\varepsilon}(z) = \mathbf{1}_{L^2(\mathbb{R})}$$

where $d\mu_{m,\varepsilon}(z)$ is the measure given by (4.7).

We close this section by mentioning the following property of these ε -CS.

Proposition 4.4. *The ε -CS obey the following thermal stability property*

$$(4.19) \quad e^{-\frac{1}{2}t(-\partial_x^2 + x^2 - \frac{1}{2})} |z; m, \varepsilon\rangle = \left(\frac{\mathcal{N}_{m,\varepsilon+t}(z)}{\mathcal{N}_{m,\varepsilon}(z)} \right)^{\frac{1}{2}} |z; m, \varepsilon + t\rangle, \quad t > 0.$$

Proof. On one hand, we write the spectral resolution of the heat operator $e^{-\frac{1}{2}t\tilde{L}}$ associated with the shifted harmonic oscillator $\tilde{L} := -\partial_x^2 + x^2 - \frac{1}{2}$ as

$$(4.20) \quad e^{-\frac{1}{2}t\tilde{L}} = \sum_{j=0}^{+\infty} e^{-\frac{1}{2}jt} |\varphi_j\rangle \langle \varphi_j|.$$

On an other hand, we rewrite the ε -CS in (4.1) as

$$(4.21) \quad |z; m, \varepsilon\rangle := (\mathcal{N}_{m,\varepsilon}(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \gamma_n^{(m)}(z) e^{-\frac{1}{2}n\varepsilon} |\varphi_n\rangle,$$

where $\gamma_n^{(m)}(z) := (\pi m! n!)^{-\frac{1}{2}} \overline{\Phi_n^m(z)}$. So that writting the action of the heat operator $e^{-\frac{1}{2}t\tilde{L}}$ on the form (4.21), we get successively

$$(4.22) \quad e^{-\frac{1}{2}t\tilde{L}} |z; m, \varepsilon\rangle = e^{-\frac{1}{2}t\tilde{L}} \left((\mathcal{N}_{m,\varepsilon}(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \gamma_n^{(m)}(z) e^{-\frac{1}{2}n\varepsilon} |\varphi_n\rangle \right)$$

$$(4.23) \quad = \sum_{j=0}^{+\infty} e^{-\frac{1}{2}tj} |\varphi_j\rangle \langle \varphi_j| \left((\mathcal{N}_{m,\varepsilon}(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \gamma_n^{(m)}(z) e^{-\frac{1}{2}n\varepsilon} |\varphi_n\rangle \right)$$

$$(4.24) \quad = (\mathcal{N}_{m,\varepsilon}(z))^{-\frac{1}{2}} \sum_{j,n=0}^{+\infty} \gamma_n^{(m)}(z) e^{-\frac{1}{2}jt - \frac{1}{2}n\varepsilon} |\varphi_j\rangle \langle \varphi_j| \varphi_n\rangle.$$

Using the orthonormality relation $\langle \varphi_j | \varphi_k \rangle = \delta_{k,j}$, this action reduces to

$$(4.25) \quad e^{-\frac{1}{2}t\tilde{L}} |z; m, \varepsilon\rangle = (\mathcal{N}_{m,\varepsilon}(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \gamma_n^{(m)}(z) e^{-\frac{1}{2}n(t+\varepsilon)} |\varphi_n\rangle$$

and it can also be written as

$$(4.26) \quad e^{-\frac{1}{2}t\tilde{L}} |z; m, \varepsilon\rangle = \left(\frac{\mathcal{N}_{m,\varepsilon+t}(z)}{\mathcal{N}_{m,\varepsilon}(z)} \right)^{\frac{1}{2}} |z; m, \varepsilon + t\rangle$$

which means that, up to a factor depending on the labelling point z , the action of the heat operator $\exp\left(-\frac{1}{2}t\tilde{L}\right)$ reproduces a similar state ε -CS, where ε is shifted by t . \square

Remark. 4.1. Eq.(4.26) means that these ε -CS satisfy a thermal stability (with respect to $\tilde{L} \geq 0$). What also make this property possible is the linearity (with respect to the integer index) of the spectrum of the Hamiltonian with HO potential. A similar fact can be found in [21] where Gazeau and Klauder introduced a real two parameters set of coherents, say $\{|J, \gamma\rangle, J \geq 0, \gamma \in \mathbb{R}\}$ associated with the discrete dynamics of a positive Hamiltonian \hat{H} . One of the requirements for the their CS was the so-called temporal stability meaning that $e^{-it\hat{H}} |J, \gamma\rangle = |J, \gamma + \omega t\rangle$, ω can be taken equal to one.

5. A CLOSED FORM FOR THE ε -CS

In this section we will establish a closed form for the constructed ε -CS and we will discuss the associated Bargmann-type integral transform.

Proposition 5.1. *Let $m \in \mathbb{Z}_+$ and $\varepsilon > 0$ be fixed parameters. Then, the wavefunctions of the ε -CS defined in (4.1) can be written in a closed form as*

$$(5.1) \quad \langle x | z; m, \varepsilon \rangle = \frac{(-1)^m \left(e^{-\frac{1}{2}\varepsilon} / \sqrt{2} \right)^m \exp \left(-\frac{1}{2}x^2 + \sqrt{2}x\bar{z}e^{-\frac{1}{2}\varepsilon} - \frac{1}{2}e^{-\varepsilon}\bar{z}^2 \right) H_m \left(x - \frac{1}{\sqrt{2}} \left(e^{\frac{1}{2}\varepsilon}z + e^{-\frac{1}{2}\varepsilon}\bar{z} \right) \right)}{(\sqrt{\pi})^{\frac{3}{2}} \sqrt{m!} \sqrt{\pi^{-1} \exp(e^{-\varepsilon}z\bar{z} - m\varepsilon) L_m^{(0)}(2(1 - \cosh \varepsilon)z\bar{z})}}$$

for every $x \in \mathbb{R}$.

Proof. We start by writing the expression of the wave function of ε -CS according to Definition (4.1) as

$$(5.2) \quad \langle x | z; m, \varepsilon \rangle = (\mathcal{N}_{m,\varepsilon}(z))^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{\overline{\Phi_n^m(z)}}{\sqrt{\sigma_{\varepsilon,m}(n)}} \varphi_n(x); \quad x \in \mathbb{R}.$$

We have thus to look for a closed form of the series

$$(5.3) \quad \mathcal{S}_z^{m,\varepsilon}(x) := \sum_{n=0}^{+\infty} \frac{\overline{\Phi_n^m(z)}}{\sqrt{\sigma_{\varepsilon,m}(n)}} \varphi_n(x).$$

To do this, we start by replacing the coefficients $\Phi_n^m(z)$ by their expression in (2.7). So that Eq.(5.3) reads

$$(5.4) \quad \mathcal{S}_z^{m,\varepsilon}(x) = \sum_{n=0}^{+\infty} \frac{e^{-\frac{1}{2}n\varepsilon}}{\sqrt{\pi n! m!}} (-1)^{n \wedge m} (m \wedge n)! |z|^{m-n} e^{-i(m-n) \arg z} L_{m \wedge n}^{(|m-n|)}(z\bar{z}) \varphi_n(x),$$

where $m \wedge n := \min(m, n)$. Next, with the help of the identity (A.7) on Laguerre polynomials, we are able to rewrite (5.4) in the following form

$$(5.5) \quad \mathcal{S}_z^{m,\varepsilon}(x) = \frac{(-1)^m \sqrt{m!}}{\sqrt{\pi}} \sum_{n=0}^{+\infty} \frac{e^{-\frac{1}{2}n\varepsilon}}{\sqrt{n!}} \bar{z}^{n-m} L_m^{(n-m)}(z\bar{z}) \varphi_n(x).$$

Making use of the explicite expression (3.7) of the eigenstates $\varphi_n(x)$, then the sum in (5.5) becomes

$$(5.6) \quad \mathcal{S}_z^{m,\varepsilon}(x) = \frac{(-1)^m \sqrt{m!}}{(\sqrt{\pi})^{\frac{3}{2}} \bar{z}^m} e^{-\frac{1}{2}x^2} \sum_{n=0}^{+\infty} \frac{\left((2e^\varepsilon)^{-\frac{1}{2}} \bar{z} \right)^n}{n!} L_m^{(n-m)}(z\bar{z}) H_n(x).$$

We now introduce the notation $\tau := (2e^\varepsilon)^{-\frac{1}{2}} \bar{z}$ in (5.6) and we will be dealing with the sum

$$(5.7) \quad G_z^{m,\varepsilon}(x) := \sum_{n=0}^{+\infty} \frac{\tau^n}{n!} H_n(x) L_m^{(n-m)}(z\bar{z}).$$

Using the integral representation of Hermite polynomials ([20], p.365):

$$(5.8) \quad H_p(x) = \frac{e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} (2iu)^p e^{-2iux} e^{-u^2} du,$$

then the sum (5.7) may be written as

$$(5.9) \quad G_z^{m,\varepsilon}(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\frac{e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} (2iu\tau)^n e^{-2iux} e^{-u^2} du \right) L_m^{(n-m)}(z\bar{z})$$

$$(5.10) \quad = \frac{e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-2iux} e^{-u^2} \left(\sum_{n=0}^{+\infty} (2iu\tau)^n \frac{1}{n!} L_m^{(n-m)}(z\bar{z}) \right) du.$$

The sum in (5.10) can also be presented as follows

$$(5.11) \quad \sum_{n=0}^{+\infty} \left(\left(\frac{1}{z} 2iu (2e^\varepsilon)^{-\frac{1}{2}} \right) z\bar{z} \right)^n \frac{1}{n!} L_m^{(n-m)}(z\bar{z}).$$

Making appeal to the following formula due to Deruyts ([23], p.142):

$$(5.12) \quad \sum_{n=0}^{+\infty} (s\alpha)^n \frac{1}{n!} L_n^{(n-m)}(s) = \frac{s^m}{m!} (\alpha - 1)^m e^{s\alpha}$$

for the parameters $\alpha = \frac{1}{z} 2iu (2e^\varepsilon)^{-\frac{1}{2}}$ and $s = z\bar{z}$ then the sum (5.11) reduces to

$$(5.13) \quad \frac{z^m}{m!} \left(iu\sqrt{2}e^{-\frac{1}{2}\varepsilon} - z \right)^m \exp \left(\sqrt{2}iue^{-\frac{1}{2}\varepsilon}\bar{z} \right).$$

Returning back to (5.10) and inserting the quantity (5.13) then (5.9) reads

$$(5.14) \quad G_z^{m,\varepsilon}(x) = \frac{\bar{z}^m e^{x^2}}{m! \sqrt{\pi}} \int_{\mathbb{R}} e^{-2iux} e^{-u^2} \left(iu\sqrt{2}e^{-\frac{1}{2}\varepsilon} - z \right)^m \exp \left(\sqrt{2}iue^{-\frac{1}{2}\varepsilon}\bar{z} \right) du.$$

We now use the binomial formula

$$(5.15) \quad \left(iu\sqrt{2}e^{-\frac{1}{2}\varepsilon} - z \right)^m = \sum_{l=0}^m \binom{m}{l} \left(iu\sqrt{2}e^{-\frac{1}{2}\varepsilon} \right)^l (-z)^{m-l}$$

and we replace it in the right hand side of (5.14). We obtain that

$$(5.16) \quad G_z^{m,\varepsilon}(x) = \frac{\bar{z}^m e^{x^2}}{m! \sqrt{\pi}} \int_{\mathbb{R}} e^{-2iux-u^2} \exp \left(\sqrt{2}iue^{-\frac{1}{2}\varepsilon}\bar{z} \right) \left(\sum_{l=0}^m \binom{m}{l} \left(iu\sqrt{2}e^{-\frac{1}{2}\varepsilon} \right)^l (-z)^{m-l} \right) du.$$

This can also be written as

$$(5.17) \quad G_z^{m,\varepsilon}(x) = \frac{\bar{z}^m e^{x^2}}{m! \sqrt{\pi}} \sum_{l=0}^m \binom{m}{l} (-z)^{m-l} \int_{\mathbb{R}} e^{-2iux-u^2} \exp \left(\sqrt{2}iue^{-\frac{1}{2}\varepsilon}\bar{z} \right) \left(iu\sqrt{2}e^{-\frac{1}{2}\varepsilon} \right)^l du.$$

Now, the integral in (5.17) :

$$(5.18) \quad I_l = \left(\frac{e^{-\frac{1}{2}\varepsilon}}{\sqrt{2}} \right)^l \int_{\mathbb{R}} \exp \left(-2iu \left(x - \bar{z} \frac{e^{-\frac{1}{2}\varepsilon}}{\sqrt{2}} \right) \right) e^{-u^2} (i2u)^l du$$

can be written by taking into account (5.8) as

$$(5.19) \quad I_l = \left(\frac{e^{-\frac{1}{2}\varepsilon}}{\sqrt{2}} \right)^l \sqrt{\pi} \exp \left(- \left(x - \bar{z} \frac{e^{-\frac{1}{2}\varepsilon}}{\sqrt{2}} \right)^2 \right) H_l \left(x - \bar{z} \frac{e^{-\frac{1}{2}\varepsilon}}{\sqrt{2}} \right).$$

Replacing (5.19) in (5.17), we arrive at

(5.20)

$$G_z^{m,\varepsilon}(x) = \frac{\bar{z}^m e^{x^2}}{m!} \left(\frac{e^{-\frac{1}{2}\varepsilon}}{\sqrt{2}} \right)^m \exp \left(- \left(x - \bar{z} \frac{e^{-\frac{1}{2}\varepsilon}}{\sqrt{2}} \right)^2 \right) \sum_{l=0}^m \binom{m}{l} \left(-\sqrt{2} z e^{\frac{1}{2}\varepsilon} \right)^{m-l} H_l \left(x - \bar{z} \frac{e^{-\frac{1}{2}\varepsilon}}{\sqrt{2}} \right).$$

Next, we apply the following identity to the last sum in (5.20) ([14], p.255):

$$(5.21) \quad \sum_{l=0}^m \binom{m}{l} (-2a)^{m-l} H_l(t) = H_m(t-a).$$

Finally, summarizing the above calculations, we arrive at the announced expression for the ε -CS in (5.1). \square

6. THE TRANSFORM $\mathcal{B}_m^\varepsilon$

Naturally, once we have obtained a closed form for the ε -CS we can look for the associated coherent states transform, say B_m^ε . In view of the definition (4.1), this transform should map the space $L^2(\mathbb{R})$ spanned by eigenstates $|\varphi_n\rangle$ of the Hamiltonian with the HO potential onto the ε -true polyanalytic space $\mathcal{A}_m^{2,\varepsilon}(\mathbb{C})$ which can be defined as the subspace of $L^2(\mathbb{C}, \pi^{-1} e^{-z\bar{z}} d\mu)$ obtained as the closure of vector space spanned by all linear combinations of the polyanalytic functions $z \mapsto (\sigma_{\varepsilon,m}(n))^{-\frac{1}{2}} \Phi_n^m(z)$ with the normalized reproducing kernel

$$(6.1) \quad K_{m,\varepsilon}(z, w) := \frac{\exp(e^{-\varepsilon} z \bar{w} - m\varepsilon)}{\sqrt{\mathcal{N}_{m,\varepsilon}(z) \mathcal{N}_{m,\varepsilon}(w)}} L_m^{(0)}((ze^{-\varepsilon} - w)(\bar{z}e^\varepsilon - \bar{w})).$$

Here, $\sigma_{\varepsilon,m}(n) = m!n!e^{n\varepsilon}$ and $\mathcal{N}_{m,\varepsilon}(z) = \exp(e^{-\varepsilon} z \bar{z} - m\varepsilon) L_m^{(0)}(2(1 - \cosh \varepsilon) z \bar{z})$.

Definition 6.1. *The Bargmann-type integral transform $B_m^\varepsilon : L^2(\mathbb{R}) \rightarrow \mathcal{A}_m^{2,\varepsilon}(\mathbb{C})$ associated with the ε -CS is defined by*

$$(6.2) \quad B_m^\varepsilon[\varphi](z) = (\mathcal{N}_{m,\varepsilon}(z))^{\frac{1}{2}} \langle \varphi | z; m, \varepsilon \rangle_{L^2(\mathbb{R})}$$

and explicitly as

$$B_m^\varepsilon[\varphi](z) := \frac{(-1)^m e^{-\frac{1}{2}m\varepsilon}}{2^{m/2} \sqrt{m!} \pi^{1/4}} \int_{\mathbb{R}} \exp \left(-\frac{x^2}{2} + \sqrt{2} x z e^{-\frac{1}{2}\varepsilon} - \frac{e^{-\varepsilon} z^2}{2} \right) H_m \left(x - \frac{e^{-\frac{1}{2}\varepsilon} \bar{z}}{\sqrt{2}} - \frac{e^{\frac{1}{2}\varepsilon} z}{\sqrt{2}} \right) \varphi(x) dx$$

for every $z \in \mathbb{C}$.

At the limit $\varepsilon \rightarrow 0^+$, we recover the coherent states transform [5] :

$$(6.3) \quad B_m^0 : L^2(\mathbb{R}) \rightarrow \mathcal{A}_m^2(\mathbb{C})$$

defined by

$$B_m^0[\varphi](z) := \frac{(-1)^m}{2^{m/2} \sqrt{m!} \pi^{1/4}} \int_{\mathbb{R}} \exp \left(-\frac{x^2}{2} + \sqrt{2} x z - \frac{z^2}{2} \right) \varphi(x) dx$$

The latter one can also be written [4] as

$$(6.4) \quad B_m^0 [\varphi] (z) \propto e^{\pi z \bar{z}} (\partial_z)^{m-1} (e^{-\pi z \bar{z}} B [\varphi] (z))$$

in terms of the transform

$$(6.5) \quad B \equiv B_0^0 : L^2 (\mathbb{R}) \rightarrow A_0^2 (\mathbb{C}) \equiv \mathcal{F} (\mathbb{C})$$

$$\varphi \mapsto B [\varphi] (z) = \pi^{-1/4} \int_{\mathbb{R}} \exp \left(-\frac{x^2}{2} + \sqrt{2} x z - \frac{z^2}{2} \right) \varphi (x) dx$$

which is the well known Bargmann transform [12] .

Appendix A

Proof. Using the orthogonality relations of the basis elements $\{\varphi_n (x)\}$ in (3.7) the scalar product in $L^2 (\mathbb{R})$ between two ε -CS can written as

$$(A1) \quad \langle z; m, \varepsilon | w; m, \varepsilon \rangle_{L^2(\mathbb{R})} = \frac{Q_\varepsilon (z, w)}{\pi m! \sqrt{\mathcal{N}_{m, \varepsilon} (z) \mathcal{N}_{m, \varepsilon} (w)}}$$

where

$$(A2) \quad Q_\varepsilon (z, w) = \sum_{n=0}^{+\infty} \frac{e^{-n\varepsilon}}{n!} \Phi_n^m (z) \overline{\Phi_n^m (w)}.$$

Recalling the explicite expression (2.7) of the of the polyanalytic coefficients, we can split the sum in (A2) into two part as

$$(A3) \quad Q_\varepsilon (z, w) = \sum_{n=0}^{m-1} e^{-n\varepsilon} n! (|z| |w|)^{(m-n)} L_n^{(m-n)} (z \bar{z}) L_n^{(m-n)} (w \bar{w}) e^{-i(m-n) \arg z} e^{i(m-n) \arg w}$$

$$+ \sum_{n=m}^{+\infty} \frac{e^{-n\varepsilon}}{n!} (m!)^2 (|z| |w|)^{(n-m)} L_m^{(n-m)} (z \bar{z}) L_m^{(n-m)} (w \bar{w}) e^{-i(m-n) \arg z} e^{i(m-n) \arg w}.$$

This quantity can also be decomposed as

$$(A4) \quad Q_\varepsilon (z, w) = Q_\varepsilon^{(<\infty)} (z, w) + Q_\varepsilon^{(\infty)} (z, w)$$

with a finite sum

$$(A5) \quad Q_\varepsilon^{(<\infty)} (z, w) := \sum_{n=0}^{m-1} e^{-n\varepsilon} n! (\bar{z} w)^{m-n} L_n^{(m-n)} (z \bar{z}) L_n^{(m-n)} (w \bar{w})$$

$$- \sum_{n=0}^{m-1} \frac{e^{-n\varepsilon}}{n!} (m!)^2 (z \bar{w})^{n-m} L_m^{(n-m)} (z \bar{z}) L_m^{(n-m)} (w \bar{w})$$

and an infinite sum

$$(A6) \quad Q_\varepsilon^{(\infty)} (z, w) := \sum_{n=0}^{+\infty} \frac{e^{-n\varepsilon}}{n!} (m!)^2 (z \bar{w})^{n-m} L_m^{(n-m)} (z \bar{z}) L_m^{(n-m)} (w \bar{w}).$$

Making appeal to the identity ([S], p.98):

$$(A7) \quad L_m^{(-k)}(t) = (-t)^k \frac{(m-k)!}{m!} L_{m-k}^{(k)}(t), \quad 1 \leq k \leq m$$

for $k = j - m$ and $t = z\bar{z}$, we can check that the finite sum $Q_\varepsilon^{(<\infty)}(z, w) = 0$. For the infinite sum in (A6), we rewrite it as

$$(A8) \quad Q_\varepsilon^{(\infty)}(z, w) = \frac{(m!)^2}{(z\bar{w})^m} \sum_{n=0}^{+\infty} \frac{1}{n!} (z\bar{w}e^{-\varepsilon})^n L_m^{(n-m)}(z\bar{z}) L_m^{(n-m)}(w\bar{w}).$$

We now apply the Wicksell-Campbell-Meixner formula ([SM], p.279):

$$(A9) \quad \sum_{n=0}^{+\infty} \frac{\zeta^n}{n!} L_l^{(n-l)}(X) L_m^{(n-m)}(Y) = e^\zeta (\zeta - Y)^{m-l} \frac{\zeta^l}{m!} L_l^{(m-l)}(-(X - \zeta)(Y - \zeta)\zeta^{-1})$$

with the notations $\zeta = e^{-\varepsilon}z\bar{w}$, $X = z\bar{z}$, $Y = w\bar{w}$ and $l = m$. With this, Eq.(A8) reduces to

$$(A10) \quad Q_\varepsilon^{(\infty)}(z, w) = m!e^{-m\varepsilon} \exp(e^{-\varepsilon}z\bar{z}) L_m^{(0)}((we^\varepsilon - z)(\bar{w}e^{-\varepsilon} - \bar{z})).$$

Finally, we replace this last expression in the right hand side of (A2) to arrive at the expression (4.3). We put $z = w$ in (4.3) and we use the condition $\langle z; m, \varepsilon | z; m, \varepsilon \rangle_{L^2(\mathbb{R})} = 1$. This allows us to obtain the expression (4.4) of the normalization factor.

Appendix B

Proof. For $x \in \mathbb{R}$, we can write successively

$$(B1) \quad \mathcal{O}_\varepsilon[\varphi](x) = \sum_{n=0}^{+\infty} e^{-n\varepsilon} \langle \varphi | \varphi_n \rangle \langle x | \varphi_n \rangle$$

$$(B2) \quad = \sum_{n=0}^{+\infty} e^{-n\varepsilon} \left(\int_{-\infty}^{+\infty} \varphi(y) \overline{\langle y | \varphi_n \rangle} dy \right) \langle x | \varphi_n \rangle$$

$$(B3) \quad = \int_{-\infty}^{+\infty} \left(\sum_{n=0}^{+\infty} e^{-n\varepsilon} \overline{\langle y | \varphi_n \rangle} \langle x | \varphi_n \rangle \right) \varphi(y) dy.$$

We now look closely at the sum

$$(B4) \quad \mathcal{G}_\varepsilon(x, y) := \sum_{n=0}^{+\infty} e^{-n\varepsilon} \overline{\langle y | \varphi_n \rangle} \langle x | \varphi_n \rangle = \sum_{n=0}^{+\infty} e^{-n\varepsilon} \varphi_n(x) \overline{\varphi_n(y)}.$$

Recalling the expression (3.7) of the $\{\varphi_n\}$ then (4.21) reads

$$(B5) \quad \mathcal{G}_\varepsilon(x, y) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(x^2+y^2)} \sum_{n=0}^{+\infty} \left(\frac{1}{2} e^{-\varepsilon} \right)^n \frac{1}{n!} H_n(x) H_n(y).$$

Eq.(B5) can be rewritten as

$$(B6) \quad \mathcal{G}_\varepsilon(x, y) = e^{-\frac{1}{2}(x^2+y^2)} K(e^{-\varepsilon}; x, y)$$

where we have introduced the kernel function

$$(B7) \quad K(\tau; x, y) := \sum_{j=0}^{+\infty} \tau^j \frac{1}{j!} H_j(x) H_j(y); \quad 0 < \tau < 1.$$

The latter can be written in a closed form by applying the Mehler formula ([14], p.252):

$$(B8) \quad K(\tau; x, y) = \frac{\pi^{-\frac{1}{2}}}{\sqrt{1-\tau^2}} \exp\left(\frac{2\tau}{1+\tau}xy - \frac{\tau^2}{1-\tau^2}(x-y)^2\right)$$

which also is the Poisson kernel for the Hermite polynomials expansion. Taking this into account, Eq.(B3) takes the form

$$(B9) \quad \mathcal{O}_\varepsilon[\varphi](x) = e^{-\frac{1}{2}x^2} \int_{-\infty}^{+\infty} \varphi(y) e^{-\frac{1}{2}y^2} K(e^{-\varepsilon}, x, y) dy.$$

We can also write the right hand side of (B9) as

$$(B10) \quad \mathcal{O}_\varepsilon[\varphi](x) = e^{-\frac{1}{2}x^2} M_\varepsilon[\varphi](x),$$

where

$$(B11) \quad M_\varepsilon[\varphi](u) = \int_{-\infty}^{+\infty} K(e^{-\varepsilon}, x, y) \varphi(y) e^{-\frac{1}{2}y^2} dy.$$

This suggests us to introduce the function

$$(B12) \quad f(y) := \varphi(y) e^{-\frac{1}{2}y^2}, y \in \mathbb{R}.$$

which statisfies

$$(B13) \quad \|f\|_{L^2(\mathbb{R}, e^{-y^2} dy)} = \|\varphi\|_{L^2(\mathbb{R})}.$$

We now apply the result of B. Muckenhoupt [Mu] who considered the Poisson integral of Hermite polynomials expansion and proved that for a function $f \in L^p(\mathbb{R}, e^{-y^2} dy)$ with $1 \leq p \leq +\infty$ the integral defined by

$$(B14) \quad A[f](\tau, x) := \int_0^{+\infty} K(\tau, x, y) f(y) e^{-y^2} dy; \quad 0 \leq \tau < 1$$

with the kernel $K(\tau, \bullet, \bullet)$ defined as given in (B8) satisfies $\lim_{\tau \rightarrow 1^-} A[f](\tau, y) = f(y)$ almost everywhere in $[0, +\infty[$, $1 \leq p \leq \infty$. We apply this result in the case $p = 2$, $A \equiv M$ and $\tau = e^{-\varepsilon}$ to obtain that $M_\varepsilon[\varphi](x) \rightarrow e^{\frac{1}{2}x^2} \varphi(x)$, *a.e.* as $\varepsilon \rightarrow 0^+$, which says that the limit $\mathcal{O}_\varepsilon[\varphi](x) = e^{-\frac{1}{2}x^2} M_\varepsilon[\varphi](x) \rightarrow \varphi(x)$, *a.e.* as $\varepsilon \rightarrow 0^+$ is valid for every $\varphi \in L^2(\mathbb{R})$. In other words, we get the limit

$$(B15) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{C}} |z; m, \varepsilon\rangle \langle z; m, \varepsilon| d\mu_{m, \varepsilon}(z) = \mathbf{1}_{L^2(\mathbb{R})}.$$

in terms of Dirac's *bra-ket* notation. This completes the proof. \square

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